Time-Domain Stability Robustness Measures for Linear Regulators

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The stability robustness aspect of linear systems is analyzed in the time domain. A bound on the perturbation of an asymptotically stable linear system is obtained to maintain stability using Liapunov matrix equation solution. The resulting bound is shown to be an improved upper bound over the ones recently reported in the literature. The proposed methodology is then extended to Linear Quadratic (LQ) and Linear Quadratic Gaussian (LQG) regulators. Examples given include comparison with an aircraft control problem previously analyzed.

Nomenclature

R^{α}	= real vector space of dimension α
δ	= Dirac delta function
$\lambda[\cdot]$	= eigenvalue of the matrix $[\cdot]$
$\rho[\cdot]$	= spectral radius of the matrix $[\cdot]$
$\sigma[\cdot]$	= singular values of the matrix $[\cdot]$
$[\cdot]_s$	= symmetric part of a matrix $[\cdot]$
$\lvert[\cdot]\rvert$	= modulus matrix = matrix with modulus entries
$\lVert[\cdot]\rVert$	= Euclidean norm of a matrix $[\cdot]$
$\lVert[\cdot]\rVert_s$	= spectral norm of a matrix $[\cdot] = \sigma_{\max}[\cdot]$

Introduction

IN the present-day applications of control systems theory and practice, one of the fundamental challenges facing a control system designer is to account for and accommodate the inaccuracies in the mathematical models of physical systems used for controller design. The inevitable presence of these errors in the model used for design eventually limits the attainable performance of the control system designs produced by either classical (frequency-domain) or modern (time-domain) control theory. Thus, it is clear that robustness is an extremely desirable, sometimes necessary, feature of any proposed feedback control design, especially for large-scale linear regulators.

For our present purposes, a robust control design is that design which behaves acceptably (i.e., satisfactorily meets the system specifications) even in the presence of modeling errors. Since the system specifications could be either in terms of stability and/or performance (regulation, time response, etc.), we can conceive two types of robustness, namely, stability robustness and performance robustness. Limiting our attention in this research to parameter errors as the type of modeling errors that may cause instability or performance degradation in the system, we formally define stability robustness as maintaining closed-loop system stability and performance robustness as maintaining a satisfactory level of performance, in the presence of modeling errors, mainly parameter variations.

This paper addresses the aspect of stability robustness in multivariable LQG regulators. Even though the aspect of performance robustness (or regulation robustness, to be more precise for the case of regulators) is equally important, it is known that performance-robustness studies assume or require stability to start with. This paper, therefore, concentrates on the stability robustness aspect. The recent published literature on this stability robustness analysis can be viewed from two perspectives, namely, frequency-domain analysis and time-domain analysis. The analysis in the frequency domain is carried out using the singular-value decomposition,[1-4] where the nonsingularity of a matrix is the criterion in developing the robustness conditions. Barrett[4] presents a useful summary and comparison of the different robustness tests available with respect to their conservatism. Bounds are obtained by Kantor and Andres[5] in the frequency domain using eigenvalue and M-matrix analysis. On the other hand, the time-domain stability-robustness analysis is presented using Liapunov stability analysis starting from Barnett and Storey,[6] Bellman,[7] Desoer et al.,[8] Davison,[9] Ackermann,[10] Franklin and Ackermann,[11] Barmish et al.,[12] and Eslami et al.[13] (in the context of robust controller design). Despite the availability of considerable analysis of the time-domain stability conditions in the above references, explicit bounds on the perturbation of a linear system to maintain stability have been reported only recently by Patel, Toda, and Sridhar,[14] Patel and Toda,[15] and Lee.[16] In Ref. 15, bounds are given for highly structured perturbations as well as for weakly structured perturbations (according to the classification given by Barrett[4]) while Lee's condition[16] treats weakly structured perturbations. Highly structured perturbations are those for which only a magnitude bound on individual elements of the perturbation matrix is known for a given model structure. Weakly structured perturbations are those for which only a spectral norm bound for the error is known.

In this paper the analysis is carried out in the time domain (along the lines of Patel and Toda[15]). A new mathematical result[17] is presented for the case of highly structured perturbations which will provide an improved upper bound over Ref. 15. An aircraft control example is presented which illustrates the "optimism" of the proposed bound compared with the one provided by Ref. 15. The analysis is then extended to the case of LQ (Linear Quadratic) and LQG (Linear Quadratic Gaussian) regulators. The usefulness of

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the proposed analysis in designing robust controllers is discussed.

Stability Robustness Measures in the Time Domain for Linear State Space Models

In this section we first present the recently available robustness measures of Patel et al.[15] Then a new robustness measure is presented for structured perturbation, and this measure is shown to be an improved measure in the sense that it is less conservative than the result of Ref. 15.

Robustness Measures According to Patel and Toda:

In Ref. 15, Patel and Toda consider the following state space description of a dynamic system,

$$\dot{x}(t) = Ax(t) + Ex(t) = (A+E)x(t) \tag{1}$$

where x is the n-dimensional state vector (R^n), A an $n \times n$ time invariant asymptotically stable matrix, and E an $n \times n$ error matrix. However, in a practical situation, one doesn't exactly know the matrix E. One may only have knowledge of the magnitude of the maximum deviation that can be expected in the entries of A. In this case (highly structured perturbation), the entries of E are such that

$$|E_{ij}| \leq \epsilon \tag{2}$$

where ϵ is the magnitude of the maximum deviation.

For this situation, it is shown in Ref. 15 that the system of Eq. (1) is stable if

$$\epsilon < \frac{\mu_P}{n} \equiv \frac{1}{n} \frac{1}{\sigma_{\max}[P]} \tag{3a}$$

where

$$\mu_P \equiv \frac{1}{\sigma_{\max}[P]} \tag{3b}$$

and P is the solution of the Liapunov matrix equation

$$A^TP + PA + 2I_n = 0 \quad (I_n \text{ is an } n \times n \text{ identity matrix}) \tag{4}$$

In what follows, we present the main mathematical result (as a new theorem[17]) which forms the basis for developing a condition for the stability of a perturbed matrix.

Main Result

Let F and E be two real matrices.

Lemma 1: If F_s is negative definite, then the matrix $F_s + E_s$ is negative definite if

$$\rho\{[E_s(F_s)^{-1}]_s\} \equiv \sigma_{\max}\{[E_s(F_s)^{-1}]_s\} < 1 \tag{5}$$

Proof: Given in Appendix A.

We now apply the above result to get an upper bound for the perturbation matrix E of system (1), assuming a highly structured perturbation.

Theorem 1: The system matrix $A+E$ of Eq. (1) is stable if

$$|E_{ij}|_{\max} = \epsilon < \frac{1}{\sigma_{\max}[|P|U_n]_s} \equiv \frac{\mu_Y}{n} \tag{6a}$$

where

$$\frac{n}{\sigma_{\max}[|P|U_n]_s} \equiv \mu_Y \tag{6b}$$

and U_n is an $n \times n$ matrix whose entries are unity, i.e., $U_{nij} = 1$ for all i, $j = 1,\ldots,n$, and P satisfies the Liapunov equation given by Eq. (4).

Example 1: We consider the same example as the one considered in Ref. 15. The nominally stable system matrix is

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \tag{7}$$

Applying the analysis of Ref. 15, and this paper, the following bounds are obtained

Patel and Toda	Yedavalli	
μ_P	μ_Y	
0.382	0.472	(8)

Thus the proposed robustness measure gives an improved upper bound.

Theorem 2: The bound $\mu_Y \geq \mu_P$ if $|P| = P$ (9)

Proof: It can be seen that

$$1/\sigma_{\max}(|P|U_n)_s \leq \sigma_{\max}(|P|U_n)$$

$$\leq \sigma_{\max}(|P|)\sigma_{\max}(U_n) \leq \sigma_{\max}(P)n \tag{10}$$

Thus

$$1/\sigma_{\max}(|P|U_n)_s \geq 1/\sigma_{\max}(P)n \tag{11}$$

$$\rightarrow n/\sigma_{\max}[(|P|U_n)_s] \geq 1/\sigma_{\max}[P] \tag{12}$$

Extension to Linear Regulators

We now extend the above analysis to the case of large-scale linear regulators having parameter variations as the modeling error.

Let us consider a continuous, linear, time-invariant system described by

$$\dot{x}(t) = Ax(t) + Bu(t) + Dw(t), \qquad x(0) = x_0 \tag{13a}$$

$$y(t) = Cx(t) \tag{13b}$$

$$z(t) = Mx(t) + v(t) \tag{13c}$$

where the state vector x is $n \times 1$, the control u is $m \times 1$, the external disturbance w is $q \times 1$, the output y (the variables we wish to control) is $k \times 1$, and the measurement vector z is $\ell \times 1$. Accordingly the matrix A is of dimension $n \times n$, B is $n \times m$, D is $n \times q$, C is $k \times n$ and M is $\ell \times n$. The initial condition $x(0)$ is assumed to be a zero-mean, Gaussian random vector with variance X_0, i.e.,

$$E[x(0)] = 0, \qquad E[x(0)x^T(0)] = X_0 \tag{14}$$

Similarly the process noise $w(t)$ and the measurement noise $v(t)$ are assumed to be zero-mean white-noise processes with Gaussian distributions having constant covariances W and V, respectively, i.e.,

$$E[w(t)] = E[v(t)] = 0 \tag{15}$$

$$E\begin{bmatrix} w(t) \\ v(t) \end{bmatrix} [w^T(\tau)\ v^T(\tau)] = \begin{bmatrix} W & 0 \\ 0 & \rho_e V_0 \end{bmatrix} \delta(t-\tau) \tag{16}$$

where ρ_e is a scalar greater than zero and $V = \rho_e V_0$.

Let the above system be evaluated for any control u by the quadratic performance index

$$J = \lim_{t\to\infty} \frac{1}{t} E\left\{\int_0^t [(y^T(\tau)Qy(\tau) + u^T(\tau)\rho_c R_0 u(\tau)]\,\mathrm{d}\tau\right\} \tag{17}$$

where scalar $\rho_c > 0$, and Q, R_0 are $(k \times k)$ and $(m \times m)$ symmetric, positive definite matrices, respectively.

For the case of a deterministic system, the following modifications in the system description are in order: 1) $Dw = 0$, $v = 0$; 2) the initial condition, $x(0) = x_0$, $x_0 x_0^T = X_0$. And the index J of Eq. (17) reads

$$J = \int_0^\infty [y^T(t) Q y(t) + u^T(t) \rho_c R_0 u(t)] \, \mathrm{d}t \tag{18}$$

If the state $x(t)$ of the stochastic system is estimated as a function of the measurements, we assume the state estimator to be of the following structure

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu + \hat{G}\hat{z}(t) \tag{19}$$

where

$$\hat{z}(t) = z(t) - M\hat{x}(t)$$

is called the measurement residual. For minimum variance requirement, the estimator of Eq. (19) is the standard Kalman filter.[18] We refer to the system presented in this section as the basic system.

Also, the following assumption is made with respect to the model described by Eqs. (13):

Assumption 1: The matrix pairs $[A,B]$ and $[A,D]$ are completely controllable, and the pairs $[A,C]$ and $[A,M]$ are completely observable.

Case I: LQ Regulators

For this case, the nominal closed-loop system matrix is given by

$$A_{\mathrm{CL}} = A + BG \tag{20a}$$

where

$$G = \frac{-1}{\rho_c} R_0^{-1} B^T K \tag{20b}$$

and

$$KA + A^T K - KB \frac{R_0^{-1}}{\rho_c} B^T K + C^T Q C = 0 \tag{20c}$$

Let ΔA, ΔB, and ΔG be the maximum modulus perturbations in the system matrices A, B, and G, respectively. Then the perturbed system matrix is

$$A_{\mathrm{CLP}} = (A + \Delta A) + (B + \Delta B)(G + \Delta G) \tag{21}$$

Design Observation 1: The perturbed LQ regulator system is stable for all perturbations in A, B, and G, in the sense of Eq. (2), if

$$\epsilon \equiv |(\Delta A + \Delta B |G| + |B + \Delta B| \Delta G)|ij_{\max} < \frac{\mu_Y}{n}$$

$$\equiv 1/\sigma_{\max}[(P|U_n)_s] \tag{22a}$$

where P satisfies

$$A_{\mathrm{CL}}^T P + P A_{\mathrm{CL}} = -2I_n \tag{22b}$$

Note that ϵ and μ_Y are functions of the control gain G.

Case 2: LQG Regulators:

For this case the optimal control for nominal values of the parameters is given by

$$u = G\hat{x} = \frac{1}{\rho_c} R_0^{-1} B^T K \hat{x} \tag{23a}$$

where

$$\dot{\hat{x}} = A\hat{x} + Bu + \hat{G}(z - M\hat{x}), \quad \hat{x}(0) = 0 \tag{23b}$$

$$= (A + BG - \hat{G}M)\hat{x} + \hat{G}z \tag{23c}$$

$$\hat{G} = \frac{1}{\rho_e} \bar{P} M^T V_0^{-1} \tag{23d}$$

and $\bar{P}$ and K satisfy the algebraic matrix Riccati equations

$$KA + A^T K - KB \frac{R_0^{-1}}{\rho_c} B^T K + C^T Q C = 0 \tag{23e}$$

$$\bar{P} A^T + A\bar{P} - \bar{P} M^T \frac{V_0^{-1}}{\rho_e} M \bar{P} + DWD^T = 0 \tag{23f}$$

The nominal closed-loop system is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BG \\ \hat{G}M & \hat{A}_c \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & \hat{G} \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \tag{24a}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \tag{24b}$$

where $\hat{A}_c = A + BG - \hat{G}M$ and the closed-loop system is asymptotically stable.

We are now interested in examining the stability robustness of the closed-loop system in the presence of parameter variations alone. Let ΔA, ΔB, ΔC, ΔM, ΔD, ΔG and $\Delta \hat{G}$ be the maximum-modulus perturbations in the system matrices, A, B, C, M, D, G, and $\hat{G}$, respectively. Then the perturbed system matrix can be written as

$$A_{\mathrm{CLP}} = \begin{bmatrix} A + \Delta A & (B + \Delta B)(G + \Delta G) \\ (\hat{G} + \Delta \hat{G})(M + \Delta M) & \hat{A}_c + \Delta \hat{A}_c \end{bmatrix} \tag{25}$$

Design Observation 2: The perturbed LQG-regulator system is stable for all perturbations in A, B, C, M, D, G, and $\hat{G}$, in the sense of Eq. (2), if

$$\frac{\mu_Y}{n} \equiv \frac{n}{\sigma_{\max}[(|P|U_n)_s]} \tag{26}$$

where

$$A_{\mathrm{CL}} = \begin{bmatrix} A & BG \\ \hat{G}M & \hat{A}_c \end{bmatrix}$$

$$E = \begin{bmatrix} \Delta A & \Delta B |G| + |B + \Delta B| \Delta G \\ |\hat{G}| \Delta M + |M + \Delta M| \Delta \hat{G} & \Delta \hat{A}_c \end{bmatrix} \tag{27}$$

and P satisfies Eq. (22b) with I_{2n} as the forcing function.

Discussion of the Design Observations

Some discussion about the implications of these design observations is now in order. First, it may be noted that the proposed stability conditions are conceptually similar to the frequency-domain results reported in Ref. 1. However, there are some interesting differences between these frequency-domain and time-domain versions. Some preliminary observations are presented in the following sections. Secondly, these design observations are useful in many ways in both the analysis and synthesis of robust controllers, which are discussed in later sections.

Comparison and Contrast Between Frequency-Domain Analysis and the Time-Domain Analysis

The main differences between the frequency-domain treatment and the time-domain treatment are as follows:

1) In the frequency-domain treatment the stability-robustness condition involves the calculation of singular values of a complex matrix at various frequencies. In the stability conditions of the time domain, no time dependence is present. Only the eigenvalues of a real symmetric matrix are to be computed.

2) In the case of frequency-domain results, the perturbations are mainly viewed in terms of gain and phase changes.[19] In the proposed time-domain analysis, the perturbations are viewed as system parameter variations with constant, fixed gains. It may be noted that in the time-domain treatment the nominally stable closed-loop matrix and the perturbed closed-loop matrix are both functions of the constant controller gains.

3) In the frequency-domain treatment, considering an uncertainty, for example, as an additive perturbation, several stability-robustness conditions can be written which do not imply each other for practical systems.[20] In the present time-domain approach, such difficulty is not present, as the perturbations are modeled as additive perturbations and yield only one robustness test.

These are some of the preliminary observations made with respect to the frequency-domain and time-domain approaches for stability robustness. Evidently further inroads have to be made in the investigation of this relationship, and this is suggested as a future-research topic. In the following section the usefulness of the proposed design observations is briefly discussed.

Usefulness of the Design Observations

The proposed design observations are helpful in many ways.

1) Given the peturbations ΔA, ΔB, and ΔM, one may determine the controller gains to achieve stability robustness.

2) This type of perturbation bound analysis can be used to compare different models and control design schemes from a stability-robustness point of view, as well as in robust controller design.

3) Finally these tests can have applications in spillover-reduction problems and sensor/actuator location problems.

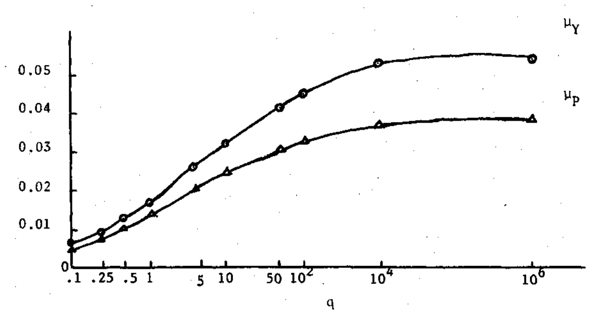

Fig. 1 Plot of μ vs q.

Table 1 Variation of μ_Y and μ_P with q

q	μ_P	μ_Y
0.1	0.0050	0.0061
0.25	0.0082	0.0093
0.5	0.0107	0.0125
1.0	0.0137	0.0164
5	0.0213	0.0272
10	0.0240	0.0322
50	0.0305	0.0420
10^2	0.0323	0.0451
10^4	0.0364	0.0530

Application to an Aircraft Control Problem

We now consider the same application example as the one considered by Patel, Toda, and Sridhar in Ref. 14. For completeness, we briefly reproduce here the mathematical model of Ref. 14.

In Ref. 14, the system chosen is the flare control of the Augmentor Wing Jet STOL Research Aircraft (AWJSRA). The purpose of the flare control is to make a smooth transition from an initial steep flight-path angle of -7.5 deg on the glide slope at an altitude of approximately 65 ft to a final smaller flight-path angle (-1 deg) more appropriate for touchdown.

The equations for the longitudinal dynamics of the AWJSRA at an airspeed of 110 ft/s and flight-path angle of -1 deg are given by

$$\dot{x} = Ax + Bu \tag{28a}$$

where

x $= [\delta v \; \delta\gamma \; \delta\theta \; \delta q \; \delta h]^T$
u $= [\delta e \; \delta n]^T$;
δv = change in airspeed, ft/s
$\delta\gamma$ = change in flight-path angle, deg
$\delta\theta$ = change in pitch angle, deg
δq = change in pitch rate, deg/s
δh = deviation from nominal altitude, ft
δe = change in elevator deflection, deg
δn = change in nozzle angle, deg

$$A = \begin{bmatrix} -0.0547 & -0.298 & -0.2639 & -0.0031 & 0.0 \\ 0.16 & -0.4712 & 0.4661 & 0.0437 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.1752 & 0.1236 & -0.1236 & -1.3 & 0.0 \\ -0.0174 & 1.92 & 0.0 & 0.0 & 0.0 \end{bmatrix} \tag{28b}$$

$$B = \begin{bmatrix} -0.00315 & -0.0943 \\ 0.0408 & 0.0224 \\ 0.0 & 0.0 \\ -1.1200 & -0.08 \\ 0.0 & 0.0 \end{bmatrix} \tag{28c}$$

The open-loop poles of the system are at 0.0, $-0.0105 \pm j0.2737$, -0.6757, and -1.129.

The performance index considered is

$$J = \int_0^\infty (x^T Q x + u^T R u) \mathrm{d}t \tag{29}$$

with $R = \text{diag}\,[16, 0.5]$ and $Q = qI_5$.

Applying the analysis of Ref. 15 and this paper, the bounds μ_Y and μ_P and their variations with q are summarized in Table 1 and Fig. 1.

Clearly μ_Y is greater than μ_P for the values of q considered, and the "optimism" of μ_Y over μ_P increases as q is increased.

Conclusions

In this paper, stability-robustness analysis is carried out in the time domain, which promises to be a viable supplement and/or alternative to the frequency-domain approach, particularly for linear state space models. An improved upper bound on the perturbation of an asymptotically stable linear system is obtained which is easy to determine numerically. Extension to LQG regulators is discussed. Some advantages of this time-domain approach are tractability of problem formulation, explict consideration of model-error information, and computational simplicity.

Appendix A

Proof of Lemma 1:

Let $\rho[(E_s(F_s)^{-1})_s] < 1$

$\rightarrow |\lambda(E_s(F_s)^{-1})_s|_{\max} < 1$

$\rightarrow |\lambda_i(E_s(F_s)^{-1})_s| < 1$

$\rightarrow 1 + \lambda_i\{[E_s(F_s)^{-1}]_s\} > 0$

$\rightarrow \lambda_i\{I + (E_s(F_s)^{-1})_s\} > 0$

$\rightarrow \lambda_i\{[I + E_s(F_s)^{-1}]_s\} > 0$

$\rightarrow [I + E_s(F_s)^{-1}]$ is positive definite

$\rightarrow [I + E_s(F_s)^{-1}][-F_s]$ has positive, real eigenvalues because 1) if A_s and B_s are positive definite, AB_s has positive real eigenvalues (Ref. [21, 6] and 2) if A_s is negative definite, $-A_s$ is positive definite and hence $-F_s$ is positive definite (Ref. [6]).

$\rightarrow -(F_s + E_s)$ has positive, real eigenvalues [because $[I + E_s(F_s)^{-1}][-F_s] = -(F_s + E_s)$]

$\rightarrow -(F_s + E_s)$ is positive definite (because $-(F_s + E_s)$ is symmetric too)

$\rightarrow (F_s + E_s)$ is negative definite

$\rightarrow (F + E)_s$ is negative definite

$\rightarrow (F + E)$ has negative real part eigenvalues

$\rightarrow (F + E)$ is stable.

Appendix B

Proof of Theorem 1: Consider

$$\dot{x} = (A + E)x(t) \tag{B1}$$

where $|E_{ij}|_{\max} = \epsilon$ (scalar) and $\Delta \equiv \epsilon U_n$, where U_n is an $n \times n$ matrix with $U_{n_{ij}} = 1$ for all $i, j = 1, 2, \ldots n$.

Let $V(x) = x^T P x > 0$ be the Liapunov function for the system in Eq. (B1) where P is the symmetric positive definite solution of

$$A^T P + PA = -2I_n \tag{B2}$$

Then

$$\dot{V}(x) = -x^T 2I_n x + x^T(E^T P + PE)x \tag{B3}$$

Now

Let $\epsilon < \dfrac{1}{\sigma_{\max}(|P|U_n)_s}$

$\rightarrow \sigma_{\max}(|P|\Delta)_s < 1$

$\rightarrow \sigma_{\max}(PE)_s < 1$

$\rightarrow \sigma_{\max}[-(PE)_s] < 1$

$\rightarrow \sigma_{\max}[(PE)_s(-I_n)^{-1}]_s < 1$

$\rightarrow [-I_n + (PE)_s]$ is negative definite (from Lemma 1)

$\rightarrow [-2I_n + E^T P + PE]$ is negative definite

$\rightarrow \dot{V}(x)$ Eq. (B3) is < 0 for all x

$\rightarrow (A + E)$ of Eq. (B1) is stable.

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